AQA A-Level Physics: Gravitational Fields – Complete Calculation Solutions

Praneel Physics

1. Calculate the gravitational force between two 5 kg masses separated by 2 m. **(P)**

Working and Answer:

Solution:

$$F = \frac{Gm_1m_2}{r^2}$$
$$= \frac{(6.67 \times 10^{-11}\ \text{N m}^2\text{kg}^{-2})(5\ \text{kg})(5\ \text{kg})}{(2\ \text{m})^2}$$
$$= \frac{1.6675 \times 10^{-9}\ \text{N m}^2}{4\ \text{m}^2}$$
$$= 4.17 \times 10^{-10}\ \text{N}$$

The gravitational force is $\boxed{4.17 \times 10^{-10}\ \text{N}}$.

2. Determine the gravitational field strength at Earth's surface (mass = 5.97×10^{24} kg, radius = 6.37×10^6 m). **(P)**

Working and Answer:

Solution:

$$\begin{aligned} g &= \frac{GM}{r^2} \\ &= \frac{(6.67 \times 10^{-11}\ \text{N m}^2\text{kg}^{-2})(5.97 \times 10^{24}\ \text{kg})}{(6.37 \times 10^6\ \text{m})^2} \\ &= \frac{3.98199 \times 10^{14}\ \text{N m}^2\text{kg}^{-1}}{4.058 \times 10^{13}\ \text{m}^2} \\ &\approx 9.81\ \text{N/kg} \end{aligned}$$

The gravitational field strength is $\boxed{9.81\ \text{N/kg}}$.

3. Calculate the gravitational potential at 10,000 km from Earth's center. **(P)**

Working and Answer:

Solution:

$$
\begin{aligned}
V &= -\frac{GM}{r} \\
&= -\frac{(6.67 \times 10^{-11}\ \text{N m}^2\text{kg}^{-2})(5.97 \times 10^{24}\ \text{kg})}{1 \times 10^{7}\ \text{m}} \\
&= -\frac{3.98199 \times 10^{14}\ \text{N m}^2\text{kg}^{-1}}{10^{7}\ \text{m}} \\
&\approx -3.98 \times 10^{7}\ \text{J/kg}
\end{aligned}
$$

The gravitational potential is $\boxed{-3.98 \times 10^{7}\ \text{J/kg}}$.

4. A satellite orbits Earth at 7,000 km from center. Calculate its orbital speed. **(P)**

Working and Answer:

Solution:

$$\begin{aligned} v &= \sqrt{\frac{GM}{r}} \\ &= \sqrt{\frac{(6.67 \times 10^{-11}\ \text{N m}^2\text{kg}^{-2})(5.97 \times 10^{24}\ \text{kg})}{7 \times 10^6\ \text{m}}} \\ &= \sqrt{5.688 \times 10^7\ \text{m}^2/\text{s}^2} \\ &\approx 7.54 \times 10^3\ \text{m/s} = 7.54\ \text{km/s} \end{aligned}$$

The orbital speed is $\boxed{7.54\ \text{km/s}}$.

5. Calculate escape velocity from Mars (mass $= 6.39 \times 10^{23}$ kg, radius $= 3.39 \times 10^{6}$ m). **(P)**

Working and Answer:

Solution:

$$
\begin{aligned}
v_{\text{esc}} &= \sqrt{\frac{2GM}{r}} \\
&= \sqrt{\frac{2(6.67 \times 10^{-11}\ \text{N m}^2\text{kg}^{-2})(6.39 \times 10^{23}\ \text{kg})}{3.39 \times 10^{6}\ \text{m}}} \\
&= \sqrt{2.514 \times 10^{7}\ \text{m}^2/\text{s}^2} \\
&\approx 5.01 \times 10^{3}\ \text{m/s} = 5.01\ \text{km/s}
\end{aligned}
$$

The escape velocity is $\boxed{5.01\ \text{km/s}}$.

6. A satellite of mass 200 kg is in circular orbit 500 km above Earth. Calculate its total energy. **(PP)**

Working and Answer:

Solution:
Step 1: Calculate orbital radius

$$\begin{aligned} r &= R_{\text{Earth}} + h = 6.37 \times 10^6 \text{ m} + 5 \times 10^5 \text{ m} \\ &= 6.87 \times 10^6 \text{ m} \end{aligned}$$

Step 2: Calculate total energy

$$\begin{aligned} E &= -\frac{GMm}{2r} \\ &= -\frac{(6.67 \times 10^{-11} \text{ N m}^2\text{kg}^{-2})(5.97 \times 10^{24} \text{ kg})(200 \text{ kg})}{2(6.87 \times 10^6 \text{ m})} \\ &= -\frac{7.966 \times 10^{16} \text{ N m}^2}{1.374 \times 10^7 \text{ m}} \\ &\approx -5.80 \times 10^9 \text{ J} \end{aligned}$$

The total energy is $\boxed{-5.80 \times 10^9 \text{ J}}$.

7. Calculate energy needed to move 50 kg from Earth's surface to 1,000 km altitude. **(PP)**

Working and Answer:

Solution:

Step 1: Identify radii

$$r_1 = 6.37 \times 10^6 \text{ m}$$
$$r_2 = 6.37 \times 10^6 \text{ m} + 1 \times 10^6 \text{ m} = 7.37 \times 10^6 \text{ m}$$

Step 2: Calculate energy difference

$$\begin{aligned}\Delta E &= GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\\ &= (6.67 \times 10^{-11})(5.97 \times 10^{24})(50)\left(\frac{1}{6.37 \times 10^6} - \frac{1}{7.37 \times 10^6}\right)\\ &= 1.992 \times 10^{16}(1.569 \times 10^{-7} - 1.357 \times 10^{-7})\\ &= 1.992 \times 10^{16}(2.12 \times 10^{-8})\\ &\approx 4.22 \times 10^8 \text{ J}\end{aligned}$$

The required energy is $\boxed{4.22 \times 10^8 \text{ J}}$.

8. Two stars (each 2×10^{30} kg) orbit with separation 1.5×10^{11} m. Find orbital period. **(PP)**

Working and Answer:

Solution:

Step 1: Calculate reduced mass system

$$\mu = \frac{M_1 M_2}{M_1 + M_2} = \frac{(2 \times 10^{30})^2}{4 \times 10^{30}} = 1 \times 10^{30}\ \text{kg}$$

Step 2: Calculate period

$$\begin{aligned} T &= 2\pi\sqrt{\frac{d^3}{G(M_1 + M_2)}} \\ &= 2\pi\sqrt{\frac{(1.5 \times 10^{11})^3}{(6.67 \times 10^{-11})(4 \times 10^{30})}} \\ &= 2\pi\sqrt{\frac{3.375 \times 10^{33}}{2.668 \times 10^{20}}} \\ &= 2\pi\sqrt{1.265 \times 10^{13}} \\ &\approx 2.24 \times 10^{7}\ \text{s} \approx 259\ \text{days} \end{aligned}$$

The orbital period is $\boxed{259\ \text{days}}$.

9. Calculate gravitational potential difference between 8,000 km and 12,000 km from Earth's center. **(PP)**

Working and Answer:

Solution:

Step 1: Calculate potentials

$$V_1 = -\frac{GM}{r_1} = -\frac{(6.67 \times 10^{-11})(5.97 \times 10^{24})}{8 \times 10^6} \approx -4.98 \times 10^7 \text{ J/kg}$$
$$V_2 = -\frac{GM}{r_2} = -\frac{(6.67 \times 10^{-11})(5.97 \times 10^{24})}{12 \times 10^6} \approx -3.32 \times 10^7 \text{ J/kg}$$

Step 2: Find difference

$$\Delta V = V_2 - V_1 \approx (-3.32 + 4.98) \times 10^7$$
$$= 1.66 \times 10^7 \text{ J/kg}$$

The potential difference is $\boxed{1.66 \times 10^7 \text{ J/kg}}$.

10. Find point between Earth and Moon where net gravitational field is zero. **(PP)**

Working and Answer:

Solution:

Step 1: Set fields equal

$$\frac{GM_E}{x^2} = \frac{GM_M}{(d-x)^2}$$
$$\frac{5.97 \times 10^{24}}{x^2} = \frac{7.35 \times 10^{22}}{(3.84 \times 10^8 - x)^2}$$

Step 2: Solve for x

$$\left(\frac{3.84 \times 10^8 - x}{x}\right)^2 = \frac{7.35 \times 10^{22}}{5.97 \times 10^{24}}$$
$$\frac{3.84 \times 10^8}{x} - 1 \approx 0.111$$
$$x \approx \frac{3.84 \times 10^8}{1.111} \approx 3.46 \times 10^8 \text{ m}$$

The zero point is $\boxed{3.46 \times 10^8 \text{ m}}$ from Earth's center.

11. A satellite is in elliptical orbit (eccentricity = 0.6, semi-major axis = 8×10^6 m). Calculate speeds at perigee and apogee. **(PPP)**

Working and Answer:

Solution:

Step 1: Find orbital distances

$$r_p = a(1-e) = 8\times10^6(1-0.6) = 3.2\times10^6 \text{ m}$$
$$r_a = a(1+e) = 8\times10^6(1+0.6) = 1.28\times10^7 \text{ m}$$

Step 2: Calculate specific angular momentum

$$\begin{aligned} h &= \sqrt{GMa(1-e^2)} \\ &= \sqrt{(6.67\times10^{-11})(5.97\times10^{24})(8\times10^6)(1-0.36)} \\ &= \sqrt{2.035\times10^{21}} \\ &\approx 4.51\times10^{10} \text{ m}^2/\text{s} \end{aligned}$$

Step 3: Compute speeds

$$v_p = \frac{h}{r_p} = \frac{4.51\times10^{10}}{3.2\times10^6} \approx 1.41\times10^4 \text{ m/s} = 14.1 \text{ km/s}$$
$$v_a = \frac{h}{r_a} = \frac{4.51\times10^{10}}{1.28\times10^7} \approx 3.52\times10^3 \text{ m/s} = 3.52 \text{ km/s}$$

The speeds are $\boxed{14.1 \text{ km/s}}$ at perigee and $\boxed{3.52 \text{ km/s}}$ at apogee.

12. Calculate total energy to assemble three 100 kg masses at vertices of 1 m equilateral triangle. **(PPP)**

Working and Answer:

Solution:

Step 1: Calculate single pair potential

$$U_{12} = -\frac{Gm_1m_2}{r} = -\frac{(6.67 \times 10^{-11})(100)(100)}{1}$$
$$= -6.67 \times 10^{-7}\ \text{J}$$

Step 2: Account for three pairs

$$U_{\text{total}} = 3U_{12} = 3(-6.67 \times 10^{-7})$$
$$= -2.00 \times 10^{-6}\ \text{J}$$

Step 3: Interpretation The negative sign indicates the system is bound. The energy required to assemble it is $\boxed{2.00 \times 10^{-6}\ \text{J}}$.

13. A spacecraft uses Hohmann transfer between Earth and Mars orbits. Calculate required Δv's. **(PPP)**

Working and Answer:

Solution:

Step 1: Orbital parameters

$$r_1 = 1.50 \times 10^{11} \text{ m (Earth)}$$
$$r_2 = 2.28 \times 10^{11} \text{ m (Mars)}$$
$$a = \frac{r_1 + r_2}{2} = 1.89 \times 10^{11} \text{ m}$$

Step 2: Initial and transfer velocities

$$v_1 = \sqrt{\frac{GM}{r_1}} = \sqrt{\frac{(6.67 \times 10^{-11})(1.99 \times 10^{30})}{1.50 \times 10^{11}}}$$
$$= 29.8 \text{ km/s}$$
$$v_{\text{transfer1}} = \sqrt{GM\left(\frac{2}{r_1} - \frac{1}{a}\right)} = 32.7 \text{ km/s}$$
$$\Delta v_1 = 32.7 - 29.8 = 2.9 \text{ km/s}$$

Step 3: Final adjustment

$$v_2 = \sqrt{\frac{GM}{r_2}} = 24.1 \text{ km/s}$$
$$v_{\text{transfer2}} = \sqrt{GM\left(\frac{2}{r_2} - \frac{1}{a}\right)} = 21.5 \text{ km/s}$$
$$\Delta v_2 = 24.1 - 21.5 = 2.6 \text{ km/s}$$

The required velocity changes are $\boxed{2.9 \text{ km/s}}$ and $\boxed{2.6 \text{ km/s}}$.

14. Calculate gravitational potential energy of uniform Earth-mass sphere. **(PPP)**

Working and Answer:

Solution:
Step 1: Potential energy of shell For a thin shell of mass dm:

$$dU = -\frac{3GM(r)dm}{5r}$$

Step 2: Integrate over entire sphere

$$U = -\frac{3}{5}\int_0^R \frac{G\left(\frac{4}{3}\pi r^3\rho\right)(4\pi r^2\rho dr)}{r}$$
$$= -\frac{16}{15}\pi^2 G\rho^2 \int_0^R r^4 dr$$

Step 3: Final calculation

$$U = -\frac{16}{15}\pi^2 G\rho^2 \frac{R^5}{5}$$
$$= -\frac{3GM^2}{5R} \approx -2.24 \times 10^{32} \text{ J}$$

The gravitational potential energy is $\boxed{-2.24 \times 10^{32} \text{ J}}$.

15. Derive orbital velocity expression and calculate speed in elliptical orbit. **(PPP)**

Working and Answer:

Solution:
Step 1: Total energy expression

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}$$

Step 2: Solve for velocity

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

Step 3: Calculate for given parameters For $r = 6 \times 10^6$ m and $a = 8 \times 10^6$ m:

$$\begin{aligned} v &= \sqrt{(6.67 \times 10^{-11})(5.97 \times 10^{24})\left(\frac{2}{6 \times 10^6} - \frac{1}{8 \times 10^6}\right)} \\ &= \sqrt{3.982 \times 10^{14}(2.92 \times 10^{-7})} \\ &\approx 7.82 \text{ km/s} \end{aligned}$$

The orbital velocity is $\boxed{7.82 \text{ km/s}}$.

16. Using Gauss's law for gravity, derive field inside/outside uniform shell and calculate at $r = R/2$ and $r = 2R$. **(PPPP)**

Working and Answer:

Solution:
Step 1: Gauss's law formulation

$$\oint \mathbf{g} \cdot d\mathbf{A} = -4\pi G M_{\text{enc}}$$

Step 2: Outside the shell (r > R)

$$g(4\pi r^2) = -4\pi G M$$
$$g = -\frac{GM}{r^2}$$

Step 3: Inside the shell (r < R)

$$M_{\text{enc}} = 0 \Rightarrow g = 0$$

Step 4: Specific calculations At $r = R/2$: $\boxed{g = 0}$
At $r = 2R$: $g = \frac{GM}{4R^2}$ $\boxed{\dfrac{GM}{4R^2}}$

17. Derive Roche limit expression and calculate for Earth and satellite ($\rho = 3000$ kg/m^3). **(PPPP)**

Working and Answer:

Solution:

Step 1: Balance tidal and self-gravity forces

$$\frac{2GM_mR}{d^3} = \frac{Gm}{R^2}$$

Step 2: Express in terms of densities

$$\frac{2G\left(\frac{4}{3}\pi R_m^3\rho_m\right)R}{d^3} = \frac{G\left(\frac{4}{3}\pi R^3\rho\right)}{R^2}$$

Step 3: Solve for distance

$$d = R\left(\frac{2\rho_m}{\rho}\right)^{1/3}$$

Step 4: Calculate for Earth

$$d = (6.37 \times 10^6)\left(\frac{2 \times 5515}{3000}\right)^{1/3}$$
$$\approx 1.8 \times 10^7 \text{ m}$$

The Roche limit is $\boxed{1.8 \times 10^7 \text{ m}}$.

18. Calculate gravitational wave luminosity of binary neutron star system. **(PPPP)**

Working and Answer:

Solution:

Step 1: Quadrupole formula

$$L_{\text{GW}} = \frac{32}{5}\frac{G^4}{c^5}\frac{(m_1 m_2)^2(m_1+m_2)}{a^5}$$

Step 2: Plug in parameters For $m_1 = m_2 = 1.4M_\odot$, $a = 20$ km:

$$L = \frac{32}{5}\frac{(6.67\times10^{-11})^4}{(3\times10^8)^5}\frac{(1.4\times1.4\times1.99\times10^{30})^2(2.8)}{(2\times10^4)^5}$$

Step 3: Compute value

$$L \approx 1.2\times10^{30}\text{ W}$$

The luminosity is $\boxed{1.2\times10^{30}\text{ W}}$.

19. Derive Mercury's precession rate and calculate per century. **(PPPP)**

Working and Answer:

Solution:

Step 1: GR correction to potential

$$\phi(r) = -\frac{GM}{r}\left(1 + \frac{h^2}{r^2c^2}\right)$$

Step 2: Calculate precession per orbit

$$\Delta\phi = \frac{6\pi GM}{a(1-e^2)c^2}$$

Step 3: Plug in Mercury's parameters

$$\Delta\phi = \frac{6\pi(6.67\times10^{-11})(1.99\times10^{30})}{(5.79\times10^{10})(1-0.206^2)(9\times10^{16})}$$
$$\approx 5.02\times10^{-7}\ \text{rad/orbit}$$

Step 4: Convert to arcseconds/century

$$\Delta\phi \approx 43\ \text{arcseconds/century}$$

The precession rate is $\boxed{43\ \text{arcseconds/century}}$.

20. Calculate tidal force on 1 kg at Earth's surface due to Moon. **(PPPP)**

Working and Answer:

Solution:

Step 1: Tidal force formula

$$F_{\text{tidal}} \approx \frac{2GM_m R_E}{d^3}$$

Step 2: Plug in values

$$F = \frac{2(6.67 \times 10^{-11})(7.35 \times 10^{22})(6.37 \times 10^6)}{(3.84 \times 10^8)^3}$$

Step 3: Compute result

$$F \approx 1.10 \times 10^{-6} \text{ N}$$

The tidal force is $\boxed{1.10 \times 10^{-6} \text{ N}}$.

21. Using GR, derive Schwarzschild radius and calculate for 10 solar mass black hole. **(PPPPP)**

Working and Answer:

Solution:

Step 1: Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \cdots$$

Step 2: Identify singularity

$$1 - \frac{2GM}{c^2 r} = 0 \Rightarrow r_s = \frac{2GM}{c^2}$$

Step 3: Calculate for $\mathbf{10}M_\odot$

$$\begin{aligned} r_s &= \frac{2(6.67 \times 10^{-11})(10 \times 1.99 \times 10^{30})}{(3 \times 10^8)^2} \\ &= \frac{2.65 \times 10^{21}}{9 \times 10^{16}} \\ &\approx 29.5 \text{ km} \end{aligned}$$

The Schwarzschild radius is $\boxed{29.5 \text{ km}}$.

22. Derive Lane-Emden equation and solve numerically for $n = 1$ mass-radius relation. **(PPPPP)**

Working and Answer:

Solution:
Step 1: Hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho\frac{GM(r)}{r^2}$$

Step 2: Mass continuity

$$\frac{dM}{dr} = 4\pi r^2\rho$$

Step 3: Polytropic relation

$$P = K\rho^{1+1/n}$$

Step 4: Dimensionless form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

Step 5: Solution for n=1

$$\theta(\xi) = \frac{\sin\xi}{\xi}$$

The solution is $\boxed{\theta(\xi) = \frac{\sin\xi}{\xi}}$.

23. Calculate Chandrasekhar limit for white dwarfs ($\mu_e = 2$). **(PPPPP)**

Working and Answer:

Solution:
Step 1: Electron degeneracy pressure

$$P \approx \frac{h^2}{m_e}\left(\frac{\rho}{\mu_e m_p}\right)^{5/3}$$

Step 2: Balance with gravity

$$\frac{GM^2}{R^4} \approx \frac{h^2}{m_e}\left(\frac{M}{\mu_e m_p R^3}\right)^{5/3}$$

Step 3: Solve for mass

$$M_{\text{Ch}} = \left(\frac{hc}{G}\right)^{3/2} \frac{1}{(\mu_e m_p)^2}$$
$$\approx 1.44 \left(\frac{2}{\mu_e}\right)^2 M_\odot$$

Step 4: Final calculation For $\mu_e = 2$:

$$M_{\text{Ch}} \approx 1.44\ M_\odot$$

The Chandrasekhar limit is $\boxed{1.44\ M_\odot}$.

24. Derive gravitational lensing deflection and calculate for light grazing Sun. **(PPPPP)**

Working and Answer:

Solution:

Step 1: Null geodesics in Schwarzschild

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2$$

Step 2: First order solution

$$\Delta\phi = \frac{4GM}{c^2 b}$$

Step 3: Solar deflection

$$\begin{aligned}\Delta\phi &= \frac{4(6.67 \times 10^{-11})(1.99 \times 10^{30})}{(3 \times 10^8)^2(6.96 \times 10^8)} \\ &= 8.48 \times 10^{-6} \text{ rad} \\ &\approx 1.75 \text{ arcseconds}\end{aligned}$$

The deflection angle is $\boxed{1.75 \text{ arcseconds}}$.

25. Using virial theorem, derive and calculate virial temperature for solar mass protostar. **(PPPPP)**

Working and Answer:

Solution:

Step 1: Virial theorem

$$2\langle K\rangle = -\langle U\rangle$$
$$3NkT = \frac{3}{10}\frac{GM^2}{R}$$

Step 2: Solve for temperature

$$T = \frac{GM\mu m_p}{5kR}$$

Step 3: Calculate for parameters

$$T = \frac{(6.67 \times 10^{-11})(1.99 \times 10^{30})(0.6)(1.67 \times 10^{-27})}{5(1.38 \times 10^{-23})(10^{12})}$$
$$\approx 1{,}928 \text{ K}$$

The virial temperature is $\boxed{1{,}928 \text{ K}}$.